VISCOUS FLUID FLOW AROUND A SPHERE

Yu. M. Petin

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Until now no methods have been found to solve the Navier–Stokes equations in their general form, as is necessary not only for investigations of such particular cases as flows with mean values of the Reynolds number but also to obtain general regularities in the theory of viscous flows.

No constraints, in principle, exist on the possibility of an iterative extension of solutions obtained for creeping flows to a higher Reynolds number range. However, up to now this possibility has not been realized although such an attempt has been made for all kinds of creeping flows for which solutions have been obtained. The reason for such a lack of success is not so much the complexity and tedium of calculating the successive approximations as the fact that the majority of solutions obtained for creeping flows are not equally suitable for all flow domains.

Indeed, a comparative estimate on the basis of the Stokes solution with totally discarded inertial terms for the Navier–Stokes equations shows that the inertial terms are Re r times greater than the viscous, which means that the Stokes linearization is competent and the Stokes solution is true only for the domain when $\mathrm{Re}\, r < 1$, i.e., for flows near the sphere surface [1]. For flows far from the surface ($r \to \infty$) the product Re r can become arbitrarily large despite any smallness of Re, in this domain the inertial terms are greater than the viscous and, therefore, the Navier–Stokes equations linearized by totally discarding the terms are inapplicable for description of flows far from a sphere. The fact of satisfaction of the boundary conditions at infinity in the Stokes solution is caused by favorable singularities inherent to the problem of the flow around a sphere. This is confirmed by the fact that it is already not possible to obtain the Stokes solution satisfying the boundary conditions at infinity for a cylinder as it turns out to be impossible to obtain the second and successive approximations for the problem of the flow around a sphere, the Whitehead paradox [2].

Oseen improved the Stokes linearization by partial conservation of the inertial terms. He obtained a solution that had no explicit constraints on Re, however, it is true only far from the sphere and does not satisfy the boundary conditions on the surface without the additional condition $\mathrm{Re} \to 0$, whereupon the velocity distribution in a potential flow was used to linearize the Navier–Stokes equations.

Proudman and Pearson [3] tried to obtain an equally suitable solution by merging the Stokes and Oseen solutions in the intermediate domain. Many successors appeared for this method but finding a solution true for higher than $\mathrm{Re} \sim 5$ was not successful. The reason here is obvious: the analytic expressions obtained in the Stokes and Oseen approximations are each true in their domain and are not correct in the merger domain and the mathematical method governing the legitimacy of the penetration of each of the solutions into the merger domain and eliminating the error that occurs here did not exist.

An attempt is made in this paper to develop a method to obtain equally suitable solutions of the complete Navier–Stokes equations for small Re values in the example of viscous fluid flow around a sphere.

The complete Navier–Stokes equations are written for the stream function in the variables r, $\mu = \cos\theta$ (the longitudinal coordinate μ is measured from the rear point) in the form

$$DD\psi = \mathrm{Re}\left(-\frac{1}{r^2}\frac{\partial\psi}{\partial\mu}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial\mu} + \frac{2}{r^3}\frac{\partial\psi}{\partial\mu} + \frac{2}{r^2}\frac{\mu}{1-\mu^2}\frac{\partial\psi}{\partial r}\right)D\psi$$
$$\left(D = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\mu^2},\quad \mathrm{Re} = \frac{U_\infty r}{\nu}\right); \tag{1}$$

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the boundary conditions are

$$\psi(1, -1 \leqslant \mu \leqslant 1) = \frac{\partial\psi(1, -1 \leqslant \mu \leqslant 1)}{\partial r}; \tag{2}$$

$$D\psi \to 0, \quad \psi \to \frac{1}{2} r^2 (1-\mu^2) \quad \text{for} \quad r \to \infty; \tag{3}$$

the axisymmetry condition is

$$\psi = D\psi = 0 \quad \text{for} \quad \mu = \pm 1. \tag{4}$$

We linearize (1) by means of the scheme

$$DD\psi = \mathrm{Re}\left(-\frac{1}{r^2}\frac{\partial\psi_i}{\partial\mu}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial\psi_i}{\partial r}\frac{\partial}{\partial\mu} + \frac{2}{r^3}\frac{\partial\psi_i}{\partial\mu} + \frac{2}{r^2}\frac{\mu}{1-\mu^2}\frac{\partial\psi_i}{\partial r}\right)D\psi. \tag{5}$$

We use the known Stokes solution for a sphere as the linearizing function ψ_i

$$\psi_i = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)(1-\mu^2). \tag{6}$$

Substituting (6) into (5), we obtain the equation

$$DD\psi = \mathrm{Re}\left[\left(1 - \frac{3}{2}\frac{1}{r} + \frac{1}{2}\frac{1}{r^3}\right)\mu\frac{\partial(D\psi)}{\partial r} + \left(\frac{1}{r} - \frac{3}{4}\frac{1}{r^2} - \frac{1}{4}\frac{1}{r^4}\right) \times (1-\mu^2)\frac{\partial(D\psi)}{\partial\mu} + \frac{3}{2}\left(\frac{1}{r^2} - \frac{1}{r^4}\right)\mu D\psi\right], \tag{7}$$

which corresponds well to physical representations on the flow pattern around a sphere. Thus for $r \to \infty$ (7) is converted into the known linearized Oseen equation which is true for any Re far from a sphere

$$DD\psi = \mathrm{Re}\left[\frac{\partial(D\psi)}{\partial r}\mu + \frac{1}{r}\frac{\partial(D\psi)}{\partial\mu}(1-\mu^2)\right], \tag{8}$$

and for small Re values is converted near the sphere surface into the Stokes approximation

$$DD\psi = 0. \tag{9}$$

Although (7) is also linear, the complexity of its convective part does not permit seeking the exact solution. To obtain the approximate solution we first find the solution of the homogeneous part of (7), i.e., the solution of (9), which we seek in the form

$$D\psi = R(r)\theta(\mu). \tag{10}$$

We have

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{(1-\mu^2)}{\theta}\frac{\partial^2\theta}{\partial\mu^2} = 0. \tag{11}$$

The solution of (11) is written in the following manner

$$D\psi = \sum_{n=1}^{\infty}\left(A_n\frac{1}{r^n} + B_n r^{n+1}\right)\left[C_n\frac{dP_n(\mu)}{d\mu} + M_n\frac{dQ_n(\mu)}{d\mu}\right](\mu^2-1) \tag{12}$$

[$P_n(\mu)$ and $Q_n(\mu)$ are Legendre polynomials of the first and second kinds]. There results from the boundary condition (3) that $B_n = 0$, while $M_n = 0$ results from (4) since $\frac{dQ_n(\mu)}{d\mu}(\mu^2 - 1) \to \infty$ as $\mu \to \pm 1$. In sum, (12) is converted into the equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\psi}{\partial\mu^2} = \sum_{n=1}^{\infty} A_n\frac{1}{r^n}\frac{dP_n(\mu)}{d\mu}(\mu^2-1), \tag{13}$$

whose solution is not difficult and the expression for the stream function is written as

$$\psi = \sum_{n=1}^{\infty}\left[\frac{1}{2(2n-1)}A_n r^{2-n} + B_n\frac{1}{r^n} + C_n r^{n+1}\right](1-\mu^2)\frac{dP_n(\mu)}{d\mu}. \tag{14}$$

Formula (14) is a solution of the Navier–Stokes equation (1) in the Stokes approximation and yields the known Stokes solution for the sphere (6) for the boundary conditions (2) and (3). The form of the expression (14) indicates the mode of obtaining an approximate solution of (7) in all.

Let us assume that (7) has successfully been converted so that the derivatives of the vorticity with respect to the longitudinal coordinate have vanished in its right side. Then the converted initial part of (7) in each iteration step could be arranged in the form $\sum_{n=1}^{\infty} f_n(r)$ $(1-\mu^2)\frac{dP_n(\mu)}{d\mu}$, analogous to (14) and a solution of (7) could therefore be obtained from the solution of several ordinary differential equations.

To realize the conversion mentioned we write

$$D\psi = \varphi(r, \mu)\exp[T(r, \mu)]. \tag{15}$$

Then (7) is converted to

$$\begin{aligned} D\varphi = {} & \left[\mathrm{Re}\left(1-\frac{3}{2}\frac{1}{r}+\frac{1}{2}\frac{1}{r^3}\right)\mu - 2\frac{\partial T}{\partial r}\right]\frac{\partial\varphi}{\partial r} + \left[\mathrm{Re}\left(\frac{1}{r}-\frac{3}{4}\frac{1}{r^2}-\right.\right. \\ & \left.\left. -\frac{1}{4}\frac{1}{r^4}\right)(1-\mu^2) - 2\frac{(1-\mu^2)}{r^2}\frac{\partial T}{\partial\mu}\right]\frac{\partial\varphi}{\partial\mu} + \left[\mathrm{Re}\left(1-\frac{3}{2}\frac{1}{r}+\frac{1}{2}\frac{1}{r^3}\right)\mu\frac{\partial T}{\partial r} + \right. \\ & + \mathrm{Re}\left(\frac{1}{r}-\frac{3}{4}\frac{1}{r^2}-\frac{1}{4}\frac{1}{r^4}\right)(1-\mu^2)\frac{\partial T}{\partial\mu} + \frac{3}{2}\mathrm{Re}\left(\frac{1}{r^2}-\frac{1}{r^4}\right)\mu - \frac{\partial^2 T}{\partial r^2} - \\ & \left. -\left(\frac{\partial T}{\partial r}\right)^2 - \frac{(1-\mu^2)}{r^2}\frac{\partial^2 T}{\partial\mu^2} - \frac{(1-\mu^2)}{r^2}\left(\frac{\partial T}{\partial\mu}\right)^2\right]\varphi. \end{aligned} \tag{16}$$

From the requirement that the partial derivatives of the vorticity with respect to the longitudinal coordinate vanish, the necessity that the brackets in the second component of the convective part of (16) vanish follows. This permits determination of the function $T(r, \mu)$

$$2\frac{1-\mu^2}{r^2}\frac{dT}{d\mu} = \mathrm{Re}\left(\frac{1}{r}-\frac{3}{4}\frac{1}{r^2}-\frac{1}{4}\frac{1}{r^4}\right)(1-\mu^2); \tag{17}$$

$$T(r, \mu) = \frac{\mathrm{Re}}{2}\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)(\mu + C). \tag{18}$$

Taking into account that μ varies between +1 and −1, and starting from the boundary condition (3) for the vorticity, it is easy to find that $C = -1$, then

$$T(r, \mu) = \frac{\mathrm{Re}}{2}\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)(\mu - 1). \tag{19}$$

Substituting (19) into (16) and neglecting terms with factors Re^n for $n > 1$ in the smallness condition for Re, we obtain an equation to determine the vorticity

$$D\varphi = \left[\mathrm{Re}\left(1+\frac{1}{2}\frac{1}{r^3}\right) - \frac{3}{2}\mathrm{Re}\frac{1}{r}\mu\right]\frac{\partial\varphi}{\partial r} + \left[\frac{3}{2}\mathrm{Re}\left(\frac{1}{r^2}-\frac{1}{r^4}\right)\mu - \frac{3}{4}\mathrm{Re}\frac{1}{r^4}\right]\varphi, \tag{20}$$

which is solved approximately by an iteration method.

We have the first approximation for $\mathrm{Re} \to 0$:

$$\varphi_1 = \sum_{n=1}^{\infty}\left(M_n r^{n+1} + K_n\frac{1}{r^n}\right)(1-\mu^2)\frac{dP_n(\mu)}{d\mu}. \tag{21}$$

Since the solution (6) exists that is true near a sphere for $\mathrm{Re} \to 0$, the boundary condition on the surface for the vorticity in (21) is determined from it while the boundary condition at infinity is determined from (3) and (15) (also as $\mathrm{Re} \to 0$). In sum, the boundary conditions are written in the form

$$\varphi_1 = (3/2)(1-\mu^2) \text{ for } r = 1; \tag{22}$$

$$\varphi_1 \to 0 \text{ for } r \to \infty \tag{23}$$

and on the basis of the solution (21)

$$\varphi_1 = \frac{3}{2}\frac{1}{r}(1-\mu^2). \tag{24}$$

An equation to determine the vorticity in a second approximation follows from (24) and (20):

$$D\varphi_2 = -\frac{3}{2}\mathrm{Re}\left(\frac{1}{r^2}+\frac{5}{4}\frac{1}{r^5}\right)(1-\mu^2) + \frac{9}{4}\mathrm{Re}\left(\frac{2}{r^3}-\frac{1}{2}\frac{1}{r^5}\right)\mu(1-\mu^2). \tag{25}$$

Since the solution of the homogeneous equation is (12) and the factors $(1 - \mu^2)$ and $\mu(1 - \mu^2)$ are the function $(1 - \mu^2)dP_n(\mu)/d\mu$ for $n = 1$ and 2, it is easy to find the general solution of (25) in a second approximation, which yields when (15) is taken into account

$$D\psi=\left[\sum_{n=1}^{\infty}\left(C_n r^{n+1}+D_n\frac{1}{r^n}\right)(1-\mu^2)\frac{dP_n(\mu)}{d\mu}+\frac{3}{4}\operatorname{Re}\left(1-\frac{3}{4}\frac{1}{r^3}\right)\times\right.$$
$$\left.\times(1-\mu^2)-\frac{9}{8}\operatorname{Re}\left(\frac{1}{r}+\frac{1}{6}\frac{1}{r^3}\right)\mu(1-\mu^2)\right]\exp\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)\frac{\operatorname{Re}}{2}(\mu-1). \tag{26}$$

This method possesses sufficiently rapid convergence, as can be illustrated by comparing the exact Oseen solution for the vorticity with that obtained by the method proposed. The Oseen formula for the stream function is

$$\psi=\frac{1}{4}\left(2r^2+\frac{1}{r}\right)(1-\mu^2)+\frac{3}{2}\frac{1}{\operatorname{Re}}(1+\mu)\left[1-\exp\frac{\operatorname{Re}}{2}r(\mu-1)\right], \tag{27}$$

and for the vorticity is

$$D\psi=\left(\frac{3}{2}\frac{1}{r}+\frac{3}{4}\operatorname{Re}\right)(1-\mu^2)\exp\frac{\operatorname{Re}}{2}r(\mu-1). \tag{28}$$

Solving the Navier–Stokes equation for the vorticity in the Oseen approximation (8) by the method proposed, we obtain, analogously to (15) and (19), that the solution of (8) must be sought in the form

$$D\psi=\varphi(r,\mu)\exp\frac{\operatorname{Re}}{2}r(\mu-1). \tag{29}$$

Substituting (29) into (8) we have

$$D\varphi=\operatorname{Re}\frac{\partial\varphi}{\partial r}. \tag{30}$$

Solving (30) under the boundary conditions (22) and (23), we find $\varphi_1=\frac{3}{2}\frac{1}{r}(1-\mu^2)$ in a first and $\varphi_2=\left(\frac{3}{2}\frac{1}{r}+\frac{3}{4}\operatorname{Re}\right)(1-\mu^2)$ in a second approximation. The vorticity is expressed in the second approximation by the formula $D\psi_2=\left(\frac{3}{2}\frac{1}{r}+\frac{3}{4}\operatorname{Re}\right)(1-\mu^2)\exp\frac{\operatorname{Re}}{2}r(\mu-1)$, which agrees completely with the exact Oseen solution (28). The third and successive approximations are the same as the second $D\psi_2=D\psi_3=D\psi_n=\left(\frac{3}{2}\frac{1}{r}+\frac{3}{4}\operatorname{Re}\right)(1-\mu^2)\exp\frac{\operatorname{Re}}{2}r(\mu-1)$, i.e., the proposed approximate method yields a solution in the second approximation that already converges to the exact value and does not change in subsequent iterations.

Returning to (26), we determine that its approximate solution can be found by first converting it by means of the relationship $\psi=F(r,\mu)\exp\left[\frac{\operatorname{Re}}{2}\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)(\mu-1)\right]$. We have

$$DF=-\operatorname{Re}\left(1+\frac{1}{2}\frac{1}{r^3}\right)(\mu-1)\frac{\partial F}{\partial r}-\operatorname{Re}\left(\frac{1}{r}-\frac{3}{4}\frac{1}{r^2}-\frac{1}{4}\frac{1}{r^4}\right)\times$$
$$\times(1-\mu^2)\frac{\partial F}{\partial\mu}+\frac{3}{4}\operatorname{Re}\frac{1}{r^4}(\mu-1)F+\operatorname{Re}\left(\frac{3}{4}-\frac{3}{16}\frac{1}{r^3}\right)(1-\mu^2)- \tag{31}$$
$$-\operatorname{Re}\left(\frac{9}{8}\frac{1}{r}+\frac{3}{16}\frac{1}{r^3}\right)\mu(1-\mu^2)+\sum_{n=1}^{\infty}\left(C_n r^{n+1}+D_n\frac{1}{r^n}\right)(1-\mu^2)\frac{dP_n(\mu)}{d\mu}.$$

The first approximation for the stream function, obtained in solving (31) when Re → 0 is written as

$$\psi=\sum_{n=1}^{\infty}\left[\left(M_n r^{n+1}+N_n\frac{1}{r^n}\right)+\left(A_n r^{n+3}+B_n r^{n+1}+C_n r^{2-n}+D_n\frac{1}{r^n}\right)\times\right.$$
$$\left.\times\exp\frac{\operatorname{Re}}{2}\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)(\mu-1)\right](1-\mu^2)\frac{dP_n(\mu)}{d\mu}.$$

From the boundary conditions of (3) at infinity, we find the constants of integration by remarking that they are independent of Re, as well as from the condition that the flow becomes potential far from the sphere by taking into account that $\exp\frac{\operatorname{Re}}{2}\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)(\mu-1)=1$ for $\mu = 1$.

TABLE 1

Re	r	$\psi(\theta=60°)$		
		Stokes solution (6)	Solution (33)	Oseen solution (27)
0,1	1,1	$5{,}454\cdot10^{-3}$	$5{,}531\cdot10^{-3}$	1,235
1,0	1,05	$1{,}384\cdot10^{-3}$	$1{,}658\cdot10^{-3}$	1,111
5,0	1,02	$5{,}606\cdot10^{-5}$	$1{,}881\cdot10^{-4}$	0,891
0,1	50	909,4	924,2	953,6
1,0	20	138,8	149,7	152,2
5,0	10	31,89	37,52	37,96
10,0	10	31,89	37,52	37,52

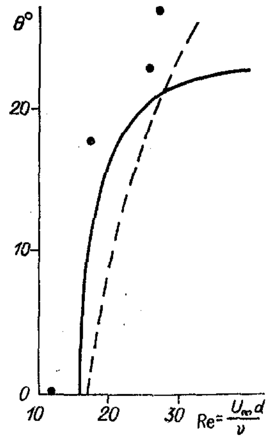

Fig. 1

In a first approximation

$$\psi_1=\left(\frac{1}{2}r^2+\frac{1}{4}\frac{1}{r}\right)(1-\mu^2)-\left[\frac{3}{4}r+\frac{9}{32}\left(r-\frac{1}{r}\right)\left(1-\frac{\mu}{r}\right)\right]\times(1-\mu^2)\exp\frac{\mathrm{Re}}{2}\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)(\mu-1). \quad (32)$$

We find the second approximation by substituting (32) into the first part of (31) and solving the equation obtained under homogeneous boundary conditions. Summing the second approximation with (32) by the method of [4], we obtain the total expression for the stream function in the r, θ variables

$$\psi=\left(\frac{1}{2}r^2+\frac{1}{4}\frac{1}{r}\right)\sin^2\theta-\left[\frac{3}{4}r+\frac{3}{16}\mathrm{Re}\left(r^2-\frac{1}{r}\right)-\right.$$
$$\left.-\frac{3}{16}\mathrm{Re}\left(r^2-\frac{1}{2}-\frac{1}{2}\frac{1}{r^2}\right)\cos\theta\right]\sin^2\theta\exp\frac{\mathrm{Re}}{2}\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)(\cos\theta-1). \quad (33)$$

The third and succeeding approximations do not differ from (33) when terms with just Re^1 are retained, i.e., even the solution for the stream function is rapidly convergent. Analysis of (33) shows that it is equally suitable for the whole flow domain.

Near the surface ($r \to 1$) (33) is converted into the Stokes formula (6), which has no constraints in Re when the condition $\mathrm{Re}\, r < 1$ is satisfied, and which, as is known, is governing and the limit of applicability of the Stokes solution. Far from the surface the computations using (33) are in good agreement with the Oseen results (27). The results illustrating the equal-suitability of the solution (33) for the whole flow domain are represented in Table 1.

The location of the point of flow separation is determined from the known condition $\partial V_\theta/\partial r = 0$ ($r = 1$), resulting in the expression

$$\cos\theta=(8+9\,\mathrm{Re}-2\sqrt{9\,\mathrm{Re}-56})/9\,\mathrm{Re}, \quad (34)$$

whose graphical display (solid curve) and comparison with numerical [5] (dashes) and experimental [6] data (points) are represented in Fig. 1.

An equation governing the appearance of a vortex behind the sphere $4 = \sqrt{9\,\mathrm{Re}\,9 - 56}$ is written from (34) for $\theta = 0$, wherefrom it is seen that a vortex first appears in the flow field for Re = 8, which is in good agreement with Re = 8.5 [5].

Shown in Fig. 2a and b are streamlines around the sphere for $\mathrm{Re} = U_\infty d/\nu = 5$ and 20, respectively (the solid lines are the results of [5] and the dashes are a computation using (33)).

Satisfactory agreement holds till at least Re = 20, where the results are closest in the rear and frontal domains and most divergent in the equatorial domain.

Figure 3 shows the location of the stationary vortex behind the sphere for $\mathrm{Re} = U_\infty d/\nu =$ 37.7 and 26.8 (lines 1 and 2), superposed for comparison and experimental points from the photographs [6]. It is seen that (33) describes the flow in the separation domain satisfactorily to $\mathrm{Re} = U_\infty d/\nu \sim 30\text{–}40$.

It is interesting to analyze the dependence of the extent of the stationary vortex on Re. The location of the downstream end of the vortex is determined for $\theta = 0$. Here

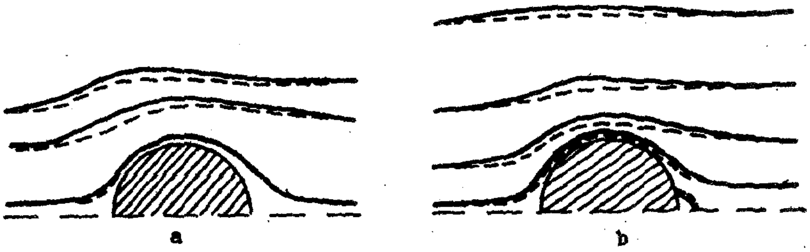

Fig. 2

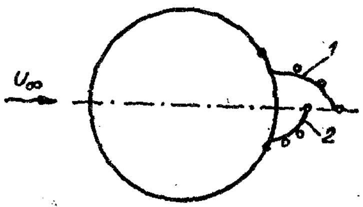

Fig. 3

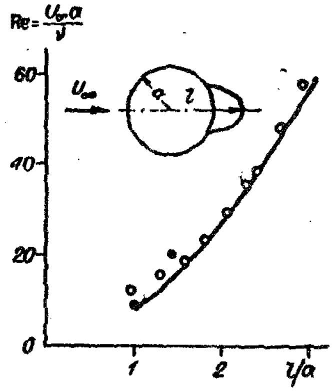

Fig. 4

$\exp \frac{\mathrm{Re}}{2}\left(r - \frac{3}{4} - \frac{1}{4}\frac{1}{r^2}\right)(\mu - 1) = 1$ and (33) is converted into the expression $\psi = (r-1)^2(1-\mu^2)\left[\frac{1}{4}\left(2 + \frac{1}{r}\right) - \frac{3}{32}\mathrm{Re}\frac{1}{r^2}\right]$ that vanishes not only on the sphere surface and along the axis of symmetry but also when the square bracket equals zero, which yields

$$\mathrm{Re} = \frac{8}{3}(2r^2 + r). \tag{35}$$

The expression obtained permits determination of the extent of the vortex for different Re values. The solution (35) is compared in Fig. 4 with experimental data [6] (open circles) and with numerical results [5] (dark points). Good agreement is conserved to $\mathrm{Re} = U_\infty d/\nu \sim 100\text{-}120$ which is close to the limit value at which a steady flow still holds.

It follows from Fig. 1 that defines the location of the stream points, from Fig. 2 and Table 1, where numerical values of the stream function are compared directly, that the solution (33) is applicable down to Re ~ 25-30. Approximately the same deduction is suggested from an analysis of the stationary vortex location behind the sphere (Fig. 3). And although an analysis of the extent of the vortex zone (Fig. 4) defines the domain of applicability to Re ~ 100-120, still no doubt is raised that the domain of applicability of the solution (33) does not exceed Re ~ 20-25, where it must be taken into account that the solution obtained has no constraints on Re for the description of flows far from the sphere, as is confirmed by the results in Table 1.

In conclusion, let us make a brief analysis of the solution (33) in order to determine the possibility of its further improvement and extension of the range of its applicability. The second component in (33) together with the exponential factor determines the whole viscous perturbation that the body around which the viscous fluid flows introduces into the free stream, the so-called "Stokeslet" of this solution. The influence of this viscous component on the solution is evidently determined mainly by the exponential factor. The viscous term in (33) vanishes and the transition to a nonvortical flow occurs when the exponential factor tends to zero and since the exponent is always negative, this will occur if the product in the exponent $\frac{\mathrm{Re}}{2}\left(r - \frac{3}{4} - \frac{1}{4}\frac{1}{r^2}\right)(\cos\theta - 1)$ will be sufficiently large, for which it is necessary that as $\mathrm{Re} \to 0$, $r \to \infty$, and therefore, a perturbation is propagated a large distance from the streamlined body in a creeping flow, i.e., the well-known fact in the theory of viscous flows is confirmed mathematically.

For Re ≫ 1 the flow rapidly goes over into the potential everywhere except the domain near the surface since as r → 1 one of the factors in the exponent $\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)$ tends to zero. Consequently, no matter how large the value of Re there is always a neighborhood near the surface where the exponent $\frac{\mathrm{Re}}{2}\left(r-\frac{3}{4}-\frac{1}{4}\frac{1}{r^2}\right)(\cos\theta-1)$ will be small, and the whole viscous permutation is concentrated in this domain, as corresponds to the physical model of the flow in boundary layer theory.

On the basis of an analysis, the deduction can be made that the method considered permits, in principle, approximate solutions to be obtained for the complete Navier–Stokes equations for a sufficiently large flow range by extending the equally suitable solution obtained to the domain of larger Re by successive approximations.

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STEADY-STATE FLOW OF A RIVULET ALONG A SURFACE UNDER THE INFLUENCE OF ACCELERATION

A. F. Tal'drik and O. P. Chernyaev

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Consideration is given to dependence of a solution for steady-state flow of a rivulet of viscous incompressible liquid on a hard flat wall on the following independent primary parameters: density ρ (kg/m^3), kinematic viscosity ν (m^2/sec), and surface tension σ (kg/sec^2) for the liquid, contact wetting angle α at the boundary of the three media, width of the main rivulet H (m) [or flow rate in the rivulet Q (m^3/sec)], field acceleration a (m/sec^2) directed along the wall. The following assumptions were made: only velocity component v (m/sec) directed along a equals zero.

A cross section of the rivulet is a region Γ bounded by a section with length H from the direction of the wall and the arc of a circle at the free surface of the rivulet. The arc of the circle and section intersect at an angle equal to α. External pressure p_0 is constant, and tangential stresses at the free surface from the direction of the external medium are ignored [1, 2].

In region Γ we find the distribution of velocities v, in particular the maximum velocity, flow rate Q_1, momentum fluxes I, and kinetic energy G in relation to the arguments enumerated above. Balance equations for the momentum and continuity for the incompressible Newtonian liquid have the form

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p/\rho + \nu\Delta\mathbf{v} + \mathbf{a},\ \nabla\cdot\mathbf{v} = 0,\ \nu = \mu/\rho. \tag{1}$$

With the assumptions made above in a coordinate system where axis OZ is directed along **a**, (1) is brought into the form $-\nabla_z p/\rho + \nu\Delta v + a = 0$, since $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$ in view of the assumption

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